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Extended Pythagoras Theorem using Triangles, and its Applications to Engineering

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ABSTRACT

Engineering is governed by mathematical laws, of which a pearl is the Pythagoras Theorem $x^2 + y^2 = z^2$ - an equation that has enabled humans to solve problems for the past millennia in all fields of science, ranging from civil to electrical engineering. In view of unlocking further benefits, the present article looks at a variant equation of the Pythagoras Theorem, namely $x^2 + xy + y^2 = z^2$. Here, it was found that just as the Pythagoras Theorem $x^2 + y^2 = z^2$ establishes an area relation between squares along the sides of a right-angled triangle, equation $x^2 + xy + y^2 = z^2$ extends this theorem by establishing an area relation between triangles along the sides of a 120-degree-angled triangle. Proofs are presented in both cases - including geometric and algebraic components - and are followed by classroom exercises, showing that both classical and extended versions of the Pythagoras Theorem are equally useful in cases of mathematics applied to engineering.

1 Introduction

1.1 Engineering and Orthogonality

We are surrounded by orthogonality, very often triggered by gravity. The simplest expression of it is a vertical wall standing on a horizontal ground. Keeping it vertical sounds like a trivial problem, but preventing vertical structures from falling has been a long lasting endeavor of humankind that is still a challenge today. Transferring information (like for instance, loads) from a vertical direction to a horizontal - which are by definition independent or decoupled - implies a diagonal path. For example, in civil engineering bridges connect an horizontal platform at length x to a vertical column of height y via intermediate diagonal beams of length z (Figure 1a).

(a)

(b)

Pictures courtesy of open source database www.pexels.com

Figure 1. Engineering examples of the application of the Pythagoras Theorem

Similarly, in aerospace engineering a flying airplane with a (ground) forward velocity vector x encountering a side wind with an orthogonal velocity vector y , together give a diagonal true air velocity vector z , towards which the aircraft is flying relative to the air (Figure 1b). Knowing the true air behavior around an aircraft is critical to allow the aerodynamic surfaces to be controlled in the proper way to guide the aeroplane to fly in the desired direction. The interrelation between these three lengths, governed by orthogonality, is by definition the very essence of the Pythagoras Theorem

$$x^2 + y^2 = z^2 \dots (1)$$

and, these examples show just how important this theorem is in solving problems in engineering. Since the Pythagoras Theorem emerged from the existence of orthogonal situations that are present in the world, would it stand to reason to think that each of the remainder angles has a theorem of its own under equivalent situations? What would they look like? This article will explore one such variant of the theorem, where the reference angle is not 90 degrees (orthogonal), but 120 degrees instead.

1.2 Babylonians and the cane against the wall

Practical problems involving the application of the Pythagoras Theorem dates back to the times of the Babylonians, where measuring the length of a cane leaning against a wall was a real challenge (Figure 2a) [1]. For the case where the wall is orthogonal to the ground (forming a right-angled triangle), the Pythagoras Theorem has proven to be the right tool to answer this riddle. But what happens to the theorem when the wall or the floor is inclined, and the new angle between them changes to 120 degrees? Let us

start by assuming the simple case where the length of the cane along the ground to the wall is 3 meters, i.e., $x=3$, and the length of the cane along the wall from the ground is 4 meters, i.e., $y=4$. The Pythagoras Theorem gives us the direct answer $3^2+4^2=5^2$, resulting in a cane length of 5 meters, i.e., $z=5$. Now imagine that the ground is in fact inclined, and the actual angle between the wall and the ground is 120 degrees (Figure 2b). Since this is no longer a right-angled triangle, the Pythagoras Theorem can no longer be applied as the necessary premise of orthogonality is no longer present.

Figure 2. Babylonian problem of a cane against a wall: (a) original right-angled triangle and (b) extended to a 120 degree-angled triangle

In this new situation, we must find another equation to determine the new length of the cane. Now, one may be tempted to use the available trigonometric function of the law of cosines. According to Euclid [2], the relation between the area of the three squares surrounding an obtuse triangle (Figure 3) is expressed as

$$AB^2 = CA^2 + CB^2 + 2(CA)(CH) \dots (2)$$

Figure 3. Obtuse triangle ABC governing the law of cosines

While $AB=z$, $CA=x$ and $CB=y$ are known, evaluation of length CH was unknown. Thus, Euclid's Elements (300 B.C.) provided the foundation that allowed for the law of cosines, with its modern expression (evolving from trigonometric tables towards its final theorem) was only presented later in the fifteenth century [3]. Replacing $CH=y\cos(\pi-\gamma)=-y\cos(\gamma)$ in Eq.(2) gives

$$z^2 = x^2 + y^2 - 2xy\cos(\gamma) \dots (3)$$

where γ is the largest internal angle, which is the law of cosines. However, this cannot be seen as a true extension of the Pythagoras theorem as it lacks the sum-of-areas geometrical proof. By expanding $\cos(\gamma)$ as a Taylor series, it is easy to see that any attempt to express it geometrically as a sum of areas is, at least, a daunting task.

$$z^2 = x^2 + y^2 - 2xy \left[1 - \frac{\gamma^2}{2!} + \frac{\gamma^4}{4!} - \frac{\gamma^6}{6!} + \dots \right]$$

This is not really what is intended in an equation that expresses the extended application of the Pythagoras Theorem to relate triangles instead of squares. Triangles because just as a square has an external angle of 90 degrees and governs the Pythagoras Theorem, a triangle has an external angle of 120 degrees and here is hypothesized to govern the new extended version of the theorem (this will be verified later in chapter 3). Corroborating this point from another perspective, just as lines that start at the corners of a square and meet at the center divide the square into four

isosceles right-angled triangles (i.e., a particular case of the triangle in Figure 2a), similarly lines starting at the corners of an equilateral triangle that meet at the center divide the triangle into three isosceles 120-degree triangles (i.e., a particular case of the triangle in Figure 2b). Placing ourselves in the shoes of Euclid, Pythagoras, and even the Babylonians, not knowing anything about cosine functions, they would probably try to find a solution that only uses what is available at the time – that is, the lengths of the side of the 120-degree-angled triangle in question. From this approach, our ideal function will be as simple as the original Pythagoras Theorem in its application, requiring only the knowledge of sides x and y to find the solution to the unknown diagonal side z .

Finally, the purpose of this article is to deliver that formula — while presenting its geometric and algebraic proof – that acts as the extended version of the Pythagoras Theorem governing the implied relations within a 120 degree-angled triangle. Indeed this is possible, as a precursor work — the publishing in 2021 of the proof of the variant of the Pythagoras Theorem using hexagons — forms a foundation, related in a parallel and complementary manner, to the proof to be presented in this article [4].

2 Pythagoras Theorem Revisited

The answer to the new riddle in Figure 2b is found by first revisiting the classical square-binding Pythagoras Theorem from the perspective of the central square theory [5].

Theorem 1 (Classical Pythagoras Theorem using Squares). *If x , y and z are real numbers, then the lengths of the sides x and y of a right-angled triangle - forming the right angle between themselves - are related to the hypotenuse of the triangle z via the sum of area of the squares annexed to each corresponding side, such that*

$$x^2 + y^2 = z^2$$

Proof. Many proofs of the Pythagoras Theorem have been presented over the years [6]. One proof of the Pythagoras Theorem is based on the aforementioned central square approach [5,7] and has the geometrical expression shown in Figure 4. It is presently shown that for the general case $x \neq y$, the split of the combined area of squares x^2 and y^2 is shown to fit completely in the square of z^2 . This is now further explained. From Figure 4a, the square EAOF associated with the side y can be decomposed into the central square KLOB', the square EPKQ side x , plus the two side rectangles PALK and QKB'F of area $x(y - x)$ each.

Figure 4. Geometric proof of the Classical Pythagoras Theorem relating areas (a) x^2 plus y^2 to (b) z^2

One small rectangle PALK plus the x^2 square EPKQ, form together a larger rectangle EALQ. Another larger rectangle is QKH'G' is formed by displacing the x^2 square OBHG to FB'H'G'. Both rectangles EALQ and QKH'G' and the central square KLOB' compose the total area of the sum x^2 and y^2 . These three elements - EALQ, QKH'G' and KLOB' - are transferred to Figure 4b. In Figure 4b, splitting these rectangles along their diagonal sides QA and QH' gives four right-angled triangles. The triangle QEA displaces to fill the space in AOD. Similarly, the triangle QG'H' is displaced to fill the space DB'H'. Concluding, all the area of the smaller square OBHG in Figure 4a of area x^2 and EAOF of area y^2 is accounted for when forming the hypotenuse square ADH'Q of area z^2 in Figure 4b .

The associated algebraic process of adding these areas, leading to equation $x^2 + y^2 = z^2$, is described as follows. The left-hand side of the intended equation shows the larger square EAOF in Figure 4a of area y^2 to be formed by a smaller square EPKQ of area x^2 added to the central square KLOB' of area $(y-x)^2$ plus two rectangles PALK and QKB'F of combined area $2x(y-x)$, resulting in

$$x^2 + 2x(y-x) + (y-x)^2 = y^2 \dots (4)$$

Likewise in Figure 4b, the right-hand side of the intended equation is formed by realizing that the square ADH'Q of area z^2 is formed by four congruent right-angled

triangles - AOD, DB'H', H'KQ and QLA - of area $\frac{xy}{2}$ each around a central square KLOB' of area $(y-x)^2$ which gives

$$4\frac{xy}{2} + (y-x)^2 = z^2 \dots (5)$$

Subtracting Eq.(4) from Eq.(5) removes the linking term - the central square $(y-x)^2$ - and Eq.(5) becomes

$$4\frac{xy}{2} - x^2 - 2x(y-x) = z^2 - y^2$$

Expanding the terms on the left, and switching sides for y^2 results in

$$2xy - 2xy + x^2 + y^2 = z^2$$

This concludes in the required Eq.(1) as

$$x^2 + y^2 = z^2$$

The proof is complete.

2.1 Particular case $x = y$

The particular case of $x=y$ is a good starting point to explore the transition between the classical Pythagoras theorem to its extended triangular version (discussed in section 3.1). Here, we begin by presenting the geometrical proof based on the central square approach [7]. This will enable a read across into the extended version, making it easier to understand the transition into the new version. Start with the square FABG inscribed in a circle (Figure 5). For the particular case $x=y$, the right-angled triangle is formed by drawing lines from points A and B to the center of the square FABG at point M (i.e., along the x- and y-axis), which is here also coincident with the center of axes O. By extending those lines MA and MB to the other corners of the square - points F and G - results in the splitting of the square FABG into four isosceles right-angled triangles (i.e., AMB, BMG, GMF and FMA).

Figure 5. Classical (square-based) Pythagoras Theorem for the particular case $x=y$

The aforementioned four triangles are placed on the outside (to the right), along the edge AB, forming the outer square ACDB with area z^2 (placing this on the right of segment AB is inline with the proof in Figure 4). The square associated with the side $y=MA$ (in this case equal to x) is obtained by drawing a line perpendicular to MF giving line FE, and perpendicular to MA giving EA, resulting in square FEAM of area x^2 . Similarly, the same can be done to the side $x=MB$ resulting in the other square MBHG also with area x^2 . These geometrical results can be achieved with any available CAD software, of which open-source examples are the programs FreeCAD [8] and Geogebra [9].

It is seen in Figure 5 that the four right-angled triangles - FEA, FMA, GMB and GHB - forming the two smaller squares FEAM and GMBH with a combined area of x^2+x^2 , are

found inside the square ACDB of area z^2 - as AM'C, CM'D, DM'B and BM'A - fulfilling the end relation

$$x^2 + x^2 = z^2 \dots (6)$$

Figure 5 is the foundation from which the particular case of the triangle version of the extended Pythagoras Theorem will be derived.

2.2 General case $y \neq x$

Moving now to the general case $x \neq y$, using as foundation the former particular case $x=y$, imagine the larger square ACDB in Figure 5 of area z^2 remains the same in size, but now rotates around the circle, as in Figure 6. The inner square FABG inscribed in the circle also rotates to a new position. The former isosceles right-angled triangle AMB in Figure 5 changes as side x is no longer the same as y , and both move away from the orthogonal axes. The new sides x and y of the triangle can be found as follows. Draw an inward line from point A parallel to the y -axis, while at the same time draw another inward line from point B parallel to the x -axis. When both lines meet at point M gives the new scalene right-angled triangle AMB (in grey). Repeating this exercise starting at all other three corners of the square - i.e., points B, G and F - gives an additional three right angle triangles (BLG, GKF and FNA), all congruent. Together they shape the central square LKNM.

Figure 6. Classical (square-based) Pythagoras Theorem for the general case $x \neq y$

Following the layout of the proof presented earlier in Figure 4, these revolving triangles are mirrored outside of the circle (to the right, including the new central square $N'K'L'M'$) within the outer square $ACDB$ of area z^2 . The square associated with the larger side $y=MA$ is obtained by drawing a line starting at M and perpendicular to MA , all the way to the circle at point I , making the segment MI . Enclosing with the successively perpendicular segments IE and EA gives the square $IEAM$ with area y^2 . Similarly, the square associated with the smaller side $x=MB$ is obtained by extending M perpendicular to MB all the way to point J at the circle forming line MJ . Enclosing with successively perpendicular segments JH' and $H'B$ gives the square $JMBH'$ with area x^2 .

Algebraically, the squares $JMBH'$ and $IEAM$ give a combined area of x^2+y^2 that has been shown in Proof 1 to be equal to the area z^2 of square $ACDB$, thus concluding in the relation

$$x^2 + y^2 = z^2$$

Figure 6 is the foundation from which the general case $x \neq y$ of the triangle version of the Pythagoras Theorem will be derived.

3 Pythagoras Theorem extended using Triangles

While the classical Pythagoras theorem operates in an environment where the lengths of the sides of the right-angled triangle are measured against an orthogonal (x- and y-axis) system, in the extended theorem the environment changes such that the lengths of the sides of the 120-degree triangle are measured against a triple (x-, y- and w-axis) system with an angular space of 60 degrees between each other.

3.1 Particular case $x = y$

Assume the length of the smaller sides x and y of the 120-degree-angled triangle are the same. Also, for this particular case $x=y$, assume the prior square FABG in Figure 5 is now replaced by an equilateral triangle DAB in Figure 7 (both inscribed in the same circle). The triangle establishing the area relation changes from a right-angled triangle to an isosceles triangle with an obtuse angle of 120 degrees. This 120-degree-angled triangle is formed in the same manner as explained in section 2.1, that is by drawing lines from points A and B to the center of the equilateral triangle DAB at point M, which is coincident with the origin point O of the triple axes. Extending an additional line parallel to the w-axis from point M to the other corner of the triangle at point D results in the splitting of the equilateral triangle DAB into three 120-degree-angled triangles – AMB, BMD and DMA. Being consistent with the Classical Pythagoras Theorem presented earlier (i.e., Figure 4-6), the aforementioned three triangles are placed to the right on the outside of the circle, along the edge AB – becoming AO'C, CO'B and BO'A – all together forming the outer equilateral triangle ACB of area $\frac{\sqrt{3}}{4}z^2$.

Figure 7. Extended (triangle-based) Pythagoras Theorem for the particular case $x=y$

The equilateral triangle EMA of area $\frac{\sqrt{3}}{4}x^2$ associated with the side $y=MA$ is obtained by drawing a line upwards starting at point M, parallel to the x-axis, all the way to the circle at point E. Enclosing it with line EA gives the desired triangle EMA. The other equilateral triangle FMB, also of area $\frac{\sqrt{3}}{4}x^2$, is obtained by drawing a line downwards starting at point M parallel to the y-axis, all the way to the circle at point F, and then enclosing it with line FB. It is observable that the combined area of these two equilateral triangles EMA and FMB together do not form the area of the outer larger equilateral triangle ACB of area $\frac{\sqrt{3}}{4}z^2$. In fact, the combined area of the triangles EMA and FMB is $\frac{\sqrt{3}}{4}(x^2 + x^2)$, which can be decomposed into four right-angled triangles (EGO, AGM, FIM and BIM), while the outer larger triangle ACB is composed of six right-angled triangles (AG'O', CG'O', CI'O', BI'O', BH'O' and AH'O'). By careful comparison, it becomes apparent that the missing additional two right-angled triangles (inside the circle) are EHM and FHM, representing a combined coupling triangle EMF of area $\frac{\sqrt{3}}{4}x^2$, that links the former two equilateral triangles EMA and FMB of area $\frac{\sqrt{3}}{4}x^2$ each.

Then it is straightforward to see that the sum of the areas $\frac{\sqrt{3}}{4}(x^2 + x^2)$ of the equilateral triangles EMA and FMB plus the coupling area $\frac{\sqrt{3}}{4}x^2$ of the linking triangle EMF (all together, the three form the total area enclosed by EAMBF) gives the six right-

angled triangles that form the equilateral triangle ACB of area $\frac{\sqrt{3}}{4}z^2$. Thus, from an algebraic point of view, the area balance for this particular case $x=y$ of the extended (triangle-based) version of the Pythagoras Theorem becomes

$$\frac{\sqrt{3}}{4}x^2 + \frac{\sqrt{3}}{4}x^2 + \frac{\sqrt{3}}{4}x^2 = \frac{\sqrt{3}}{4}z^2 \dots (7)$$

Where the coefficient $\frac{\sqrt{3}}{4}$ can be conveniently cancelled on either side, resulting in

$$x^2 + x^2 + x^2 = z^2$$

3.2 General case $y \neq x$

Imagine the outer equilateral triangle ACB of area $\frac{\sqrt{3}}{4}z^2$ in Figure 7 remains the same size, but now rotates clockwise around the circle to the position shown in Figure 8. The inner mirrored equilateral triangle DAB also of area $\frac{\sqrt{3}}{4}z^2$ (inside the circle) displaces along its edges, making the reference 120-degree-angled triangle change size, shape and position. For the general case $x \neq y$, the now scalene 120-degree-angled triangle AMB in Figure 8, with the length of its sides x and y becoming different from each other, moves away from the corresponding axes. These new sides of the 120-degree-angled triangle can be found as before. Draw a line from point A downwards parallel to the y -axis, while at the same time draw a line upwards from point B parallel to the x -axis. When both lines meet at point L forms the new 120-degree-angled triangle AMB (filled in grey). Repeating this exercise starting at the other two corners of the equilateral triangle DAB - i.e., points B and D - gives an additional two scalene 120-degree-angled triangles BKD and DLA that (with AMB) are congruent. That is, imagine a line emerging simultaneously from each of the three corners of the equilateral triangle DAB, in an inward direction and parallel to each of the x -, y - and w -axis. These lines will eventually cross each other at points K, L and M - forming the segments AM, BK and CL. Since these three lines are parallel to each axis, they form an angle of 60 degrees between themselves. Since they do not meet in a single point (as they did in point $M \equiv O$ in Figure 8), three lines forming 60 degrees amongst each other will inherently form a triangle. Due to symmetry between all three axis, all the three lines AM, BK and DL are the same, but symmetrically rotated by 120 degrees around point O, which means they cross each other at the same location, pointing to an equal distance between the ends of the segments (for example, point M for segment AM) and the end of the segment (in this example, point A). Hence, segments LM, MK and KL have the same length, form internally 60 degrees amongst each other (or 120 degrees externally), and compose the sides of a triangle. The end conclusion is that

these segments, all together under the defined constraints, form amongst themselves (internally to DAB) a central equilateral triangle KML.

Figure 8. Extended (triangle-based) Pythagoras Theorem for the general case $x \neq y$

Following the prior approach from Figure 5-7, the three revolving triangles - AMB, BKD and DLA - are mirrored outside the circle (to the right) - as AL'C, CK'B and BM'A - within the outer equilateral triangle ACB of area $\frac{\sqrt{3}}{4}z^2$. Together, they form internally the mirrored central triangle M'K'L'. The equilateral triangle EMA associated with the larger side $y=MA$ is obtained by drawing a line upwards starting at point M and inline with the x-axis all the way to the circle at point E to form segment ME. Enclosing with line EA gives the equilateral triangle EMA of area $\frac{\sqrt{3}}{4}y^2$. Similarly, the equilateral triangle FMB associated with the smaller side $x=MB$ is obtained by extending from point M downwards inline with the y-axis all the way to the circle at point F, forming the segment MF. Enclosing with line FB gives the equilateral triangle FMB of area $\frac{\sqrt{3}}{4}x^2$. The coupling triangle EMF is obtained by connecting point E to F, giving a 120-degree-angled triangle of area $\frac{\sqrt{3}}{4}xy$, that links together the aforementioned two equilateral triangles EMA of area $\frac{\sqrt{3}}{4}y^2$ and FMB of area $\frac{\sqrt{3}}{4}x^2$.

Theorem 2 (Extended Pythagoras Theorem using Triangles). *If x , y and z are real numbers, then the lengths of sides x and y of a 120-degree-angled triangle - forming the 120-degree angle between themselves - are related to the length of the longest side of the triangle z via the sum of the areas of the equilateral triangles that are annexed to each corresponding side, plus the area of a coupling 120-degree-angled triangle, resulting in*

$$\frac{\sqrt{3}}{4}x^2 + \frac{\sqrt{3}}{4}xy + \frac{\sqrt{3}}{4}y^2 = \frac{\sqrt{3}}{4}z^2 \dots (8)$$

Proof. The geometrical expression of the extended version of the Pythagoras Theorem using triangles is shown in Figure 9, which emerges as an intentional evolution from the Classical Pythagoras Theorem shown previously in Figure 4. For the general case of $x \neq y$, the splitting of the area composed of the smaller equilateral triangles $\frac{\sqrt{3}}{4}x^2$ and $\frac{\sqrt{3}}{4}y^2$ plus the coupling 120-degree-angled triangle (Figure 9a) is shown to fit the totality of the area of the equilateral triangle $\frac{\sqrt{3}}{4}z^2$ (Figure 9b). This is now further explained.

Figure 9a shows the reference 120-degree-angled triangle $AB'B$ that replaces here the traditional right-angled triangle. At the bottom, along the side $x=B'B$ of the 120-degree-angled triangle $AB'B$, there is the equilateral triangle $FB'B$ of area $\frac{\sqrt{3}}{4}x^2$, which replaces the traditional square x^2 found in the classical theorem. At the top of Figure 9a, along the side $y=AB'$, there is the equilateral triangle $EB'A$ of area $\frac{\sqrt{3}}{4}y^2$ that can be decomposed into the central triangle $D'A'B'$ of area $\frac{\sqrt{3}}{4}(y-x)^2$, another equilateral triangle HJI associated with side $x=HJ$ of area $\frac{\sqrt{3}}{4}x^2$, plus two side skewed rectangles $EHJD'$ and $IAA'J$. These two equilateral triangles $EB'A$ of area $\frac{\sqrt{3}}{4}y^2$ and $FB'B$ of area $\frac{\sqrt{3}}{4}x^2$ are coupled by a 120-degree-angled triangle $EB'F$ of area $\frac{\sqrt{3}}{4}xy$. To the right of the reference 120-degree-angled triangle $AB'B$, there is the equilateral triangle ACB associated with side $z=AC$ of area $\frac{\sqrt{3}}{4}z^2$.

Figure 9. Geometric proof of the Extended Pythagoras Theorem relating triangle areas (a) $\sqrt{3}/4x^2$, $\sqrt{3}/4xy$ plus $\sqrt{3}/4y^2$ to (b) $\sqrt{3}/4z^2$

This is repeated and dealt separately in Figure 9b, being composed of three congruent 120-degree-angled triangles - $AA'C$, $CD'B$ and $BB'A$ - plus the central triangle $A'D'B'$. In Figure 9a, the equilateral triangle $FB'B$ of area $\frac{\sqrt{3}}{4}x^2$ ($BB'F$ in Figure 9a), the two small skewed rectangles $EHJD'$ and $IAA'J$ ($LKMF$ and $B'D'KL$ in Figure 9b) plus the equilateral triangle HJI of area $\frac{\sqrt{3}}{4}x^2$ ($D'CM$ in Figure 9b) form together in Figure 9b a larger skewed rectangle $BD'CF$ of sides $x=BF$ and $y=FC$. Splitting the larger skewed rectangle $BD'CF$ (in Figure 9b) along its diagonal BC gives two congruent 120-degree-angled triangles. One fills the space in $BD'C$ and the other BFC is displaced to fill the space in $AA'C$. The 120-degree-angled triangle in Figure 9a $EB'F$ becomes $AB'B$ in Figure 9b. The area of a 120-degree-angled triangle (like in Figure 9a $AB'B$ or in Figure 9b $AA'C$) is found in the following manner. It is seen at the bottom of Figure 9b that the larger skewed rectangle $BD'CF$ of sides $x=BF$ and $y=FC$ is formed by two smaller equilateral triangles $BB'F$ and $D'CM$ and two smaller skewed rectangles $LKMF$ and $B'D'KL$. Dividing the rectangle along its diagonal BC gives two congruent 120-degree-angled triangles $BD'C$ and BFC that have an area, by definition, of half the base length $y=FC$ multiplied by its projected height $NF=\frac{\sqrt{3}}{2}x$, or

$$\text{Area of Triangle } AB'B = \frac{1}{2}y\left(\frac{\sqrt{3}}{2}x\right) = \frac{\sqrt{3}}{4}xy$$

Concluding, all the area in Figure 9a of the smaller equilateral triangle FB'B associated with side $x=B'B$ of area $\frac{\sqrt{3}}{4}x^2$, the larger equilateral triangle EB'A associated with side $y=B'A$ of area $\frac{\sqrt{3}}{4}y^2$, plus the coupling 120-degree-angled triangle EB'F of area $\frac{\sqrt{3}}{4}xy$ is accounted for in forming the longest side equilateral triangle ACB of area $\frac{\sqrt{3}}{4}z^2$ in Figure 9b. The associated algebraic process of adding areas, leading to the extended version of the Pythagoras Theorem equation, follows below from the aforementioned description. Both Figure 9a and 9b represent independently in terms of area addition an equation, and both incorporate a central triangle that link both equations to provide the desire solution. In Figure 9a, the larger equilateral triangle EB'A of area $\frac{\sqrt{3}}{4}y^2$ is composed of a smaller equilateral triangle HJI of area $\frac{\sqrt{3}}{4}x^2$, two smaller skewed rectangles EHJD' and IAA'J of combined area $2\frac{(y-x)}{2}\left(\frac{\sqrt{3}}{2}x\right)$, plus the central triangle D'A'B' of area $\frac{\sqrt{3}}{4}(y-x)^2$, resulting in

$$\frac{\sqrt{3}}{4}x^2 + 2\frac{(y-x)}{2}\left(\frac{\sqrt{3}}{2}x\right) + \frac{\sqrt{3}}{4}(y-x)^2 = \frac{\sqrt{3}}{4}y^2$$

that re-arranged gives

$$\frac{\sqrt{3}}{4}x^2 + 2\frac{\sqrt{3}}{4}x(y-x) + \frac{\sqrt{3}}{4}(y-x)^2 = \frac{\sqrt{3}}{4}y^2 \dots (9)$$

This forms one equation, and the second equation is retrieved from Figure 9b. This is now described. The equilateral triangle ACB in Figure 9b of area $\frac{\sqrt{3}}{4}z^2$ is composed of three congruent 120-degree-angled triangles - AA'C, CD'B and BB'A - of area $\frac{\sqrt{3}}{4}xy$ each, revolving around a central triangle A'D'B' of area $\frac{\sqrt{3}}{4}(y-x)^2$, giving

$$3\frac{\sqrt{3}}{4}xy + \frac{\sqrt{3}}{4}(y-x)^2 = \frac{\sqrt{3}}{4}z^2 \dots (10)$$

Subtracting Eq.(9) from Eq.(10) removes the linking term - the central triangle $\frac{\sqrt{3}}{4}(y-x)^2$ - and Eq.(10) becomes

$$3\frac{\sqrt{3}}{4}xy - \frac{\sqrt{3}}{4}x^2 - 2\frac{\sqrt{3}}{4}x(y-x) = \frac{\sqrt{3}}{4}z^2 - \frac{\sqrt{3}}{4}y^2$$

Expanding and simplifying the terms on the left, and placing $\frac{\sqrt{3}}{4}y^2$ on the other side gives

$$\frac{\sqrt{3}}{4}x^2 + \frac{\sqrt{3}}{4}xy + \frac{\sqrt{3}}{4}y^2 = \frac{\sqrt{3}}{4}z^2 \dots (11)$$

Thus, completing the proof.

Conveniently removing the coefficient $\frac{\sqrt{3}}{4}$ results in the required relation

$$x^2 + xy + y^2 = z^2 \dots (12)$$

The initial challenge raised in Figure 2b regarding the extended Babylonian problem of the length of a cane against an inclined slope in Figure 2b (or an inclined wall) such that they form a 120-degree angle is answered by using Eq.(12) as $z = \sqrt{y^2 + xy + x^2}$, where the key difference with respect to the original orthogonal problem in Figure 2a having a solution $z = \sqrt{y^2 + x^2}$ is the coupling term xy . While Eq.(12) is a more convenient way to write Eq.(11) as it speeds up calculations (and indeed the relative proportion between areas is expressed correctly), one must remember that in absolute terms the coefficient $\frac{\sqrt{3}}{4}$ needs to be accounted for to express the actual area of the triangle $\frac{\sqrt{3}}{4}x^2$ that is not x^2 , and thus the more complete (but less convenient) way to express the extended theorem is Eq.(11). Keeping this coefficient in mind, one can algebraically apply Eq.(12) as a matter of convenience.

As a summary, the classical Pythagoras Theorem interrelating squares via a right-angled triangle is shown in Figure 10a, while the extended Pythagoras Theorem interrelating triangles via a 120-degree-angled triangle is shown in Figure 10b. Building the extended version of the Pythagoras Theorem is almost as straight forward as the classical version, and only requires one to draw three equilateral triangles, one for each of the three sides of the 120-degree-angled triangle, and draw a line interconnecting the upper left edge of the y-bounded equilateral triangle to the lower left edge of the x-bounded equilateral triangle, to form the coupling area, as shown in Figure 10b.

Figure 10. Back-to-back Pythagoras Theorems: (a) Classical using squares and (b) Extended using triangles

As a conclusion, while the classical orthogonal Pythagoras Theorem interconnecting squares via a right-angled triangle (Figure 10a) is given by

$$x^2 + y^2 = z^2$$

The new extended Pythagoras Theorem interconnection of triangles via a 120 degree-angled triangle (Figure 10b) is given by

$$\begin{aligned} \frac{\sqrt{3}}{4}x^2 + \frac{\sqrt{3}}{4}xy + \frac{\sqrt{3}}{4}y^2 &= \frac{\sqrt{3}}{4}z^2 \\ \implies x^2 + xy + y^2 &= z^2 \end{aligned}$$

This result can be validated via the law of cosines by replacing the angle $\gamma=120^\circ$ into Eq.(3), repeated below for convenience

$$z^2 = x^2 + y^2 - 2xy\cos(\gamma)$$

which results in the substitution $\cos(120\text{deg}) = -\frac{1}{2}$ giving

$$z^2 = x^2 + y^2 - 2xy\left(-\frac{1}{2}\right)$$

Thus arriving at the same result as Eq.(12)

$$x^2 + xy + y^2 = z^2$$

Of particular interest is the coupling term xy that emerges as a key difference between the classical and the extended version of the Pythagoras Theorem. Its physical meaning will be explained in the following exercises.

4 Exercises

Two classroom examples are presented with both theorems being applied, first relating dimensions of beams useful in bridge building for civil engineering, and second relating number of coils and their angular clocking used in the design of alternating current generators and motors for electrical engineering.

4.1 Civil Engineering

The following exercise, involving beams in a bridge (Figure 11), illustrates how both theorems are equally helpful. While this is a straight forward example, it is worthwhile recording it for the sake of those students that are beginners to the fundamental application of the Pythagoras Theorem, and its extensions. Imagine that one wants to know the distance between the arc and the platform at two different locations. At location A, the angle between the beams is orthogonal and the beams are $x=0.7m$ and $y=1m$ in length, and at location B the angle between the beams is 120 degrees and the beams are $x=1.5m$ and $y=2m$ in length. What are the respective distances? The answer is obtained by applying both the classical and extended Pythagoras Theorem. For the orthogonal case, the classical Eq.(1) gives

$$z = \sqrt{x^2 + y^2} = \sqrt{0.7^2 + 1^2} = 1.220 \dots$$

While for the case where both beams form 120 degrees, the extended Eq.(12) results in

$$\begin{aligned} z &= \sqrt{x^2 + xy + y^2} = \\ &= \sqrt{1.5^2 + (1.5)(2) + 2^2} = 3.041 \dots \end{aligned}$$

Picture courtesy of open source database www.pexels.com

Figure 11. Classical and Extended Pythagoras Theorems applied to determine lengths in a bridge

4.2 Electrical Engineering

Systems inherently possessing orthogonality, like the pendulum, invoke the usage of the classical Pythagoras Theorem to determine its dynamical behavior. Similarly, systems operating based on 120-degree angles, like the electrical alternating current system, naturally invoke the usage of the extended Pythagoras Theorem using triangles, presented earlier in this article. As another associated example, three-phase electric power systems — popularly employed in electrical engineering — operate by setting three alternating currents that are out of phase by 120 degrees, while having the same frequency [10,11].

Alternating current generators and motors operate by placing coils around a rotating magnet. The electromagnetic field generated by the permanent magnet at the center is fixed, having a north and south pole where intensity is maximum. A gradual change in intensity is observed as one rotates the magnet from one pole to the other. This change in electromagnetic field is converted by a coil into alternating current. Naturally, having more coils produces more alternating current, but they have to be grouped around the magnet in a harmonious way. This is where the classical square-based

Pythagoras Theorem and the new extended triangle-based version are useful. Figure 12 shows an alternating current motor/generator (the cut-out is for illustration purposes) with a zoom (on the right) to one of its sectors composed of two coils angular-spaced by 90° (assuming a quadruple coil system disposed as a square) with a rotating magnet at the center. How does one perceive the Pythagoras Theorem in a physical sense in an alternating generator or motor?

Figure 12. Classical Pythagoras Theorem application to a sector of an electric motor/generator with 90° -spaced coils

Imagine the peak electromagnetic field of the magnet is represented by the area of the square z^2 , and is naturally fixed and independent of angular position. As the magnet rotates, the intensity of the field going through coil A - here represented by the area of square x^2 - is decreasing (in blue), while that in coil B - here represented by the area of the square y^2 - is increasing (in purple). Therefore, the Pythagoras Theorem states that in an orthogonal system (i.e., when coils are clocked by 90 degrees), for every angle the magnet rotates, the drop in intensity in coil A (i.e., drop in area x^2) is compensated by a raise in intensity in coil B (i.e., raise in area y^2), or

$$x^2 + y^2 = z^2$$

Why is this important? Because the intensity of the magnetic field perceived by the system composed of the sum of the two coils is always the same, equal to the peak intensity of the magnet (i.e., the area z^2), independently of the angular position of the magnet. That is, there is a continuous constant power generation (or loading in the case of a motor) as the magnet turns.

Exercise 1. Imagine the magnet has a hypothetical intensity of 1, and coil A is recording an intensity $x^2=0.7^2=0.49$. How much intensity is being recorded by coil B located in a clockwise position at 90° , and how much is being wasted?

Solution. Assume $z=1^2$ and $x^2=0.7^2=0.49$. Applying the classical Pythagoras Theorem gives

$$0.7^2 + y^2 = 1^2$$

Resulting in,

$$y = \sqrt{1^2 - 0.7^2} = 0.714 \dots$$

This means that coil A sees an intensity of $x^2=0.49$ or 49%, while coil B sees $y^2=0.51$ or 51%. Since the sum of the two gives 100% of the intensity of the magnet, nothing is being wasted. Figure 13 shows an alternating current motor/generator (the cut-out is for illustration purposes) with a zoom (on the right) to one of its sectors composed of two coils angular-spaced by 120° (assuming a triple-coil system disposed as a triangle) with a rotating magnet at the center.

Figure 13. Extended Pythagoras Theorem application to a sector of an electric motor/generator with 120° -spaced coils

This time, imagine the peak electromagnetic field of the magnet is represented by the equivalent area of the equilateral triangle z^2 , and is naturally fixed and independent of angular position. As before, when the magnet rotates, the intensity of the field going through coil A - here represented by the area of the square x^2 - is decreasing (in blue), while that in coil B - here represented by the area of the square y^2 - is increasing (in

purple). However, because the coils are more angular-spaced apart, the rate at which the intensity changes between them will be different. The extended Pythagoras Theorem states that in a triple-phase system (i.e., when coils that are clocked by 120 degrees), for every angle the magnet rotates, the drop in intensity in coil A (i.e., drop in area x^2) is compensated by a raise in intensity in coil B (i.e., raise in area y^2) plus a part of this intensity (in yellow) that is not perceived by neither coil A and B (i.e., the coupling area xy), giving

$$x^2 + xy + y^2 = z^2$$

Again, why is this important? Because the total intensity of the magnetic field perceived by the system composed of these two coils drops (below the peak magnet intensity), as the magnet moves away from coil A until a minimum is reached halfway between the two coils, identifying a loss in the system. Beyond this, the system's perceived intensity rises again until it reaches once more the peak intensity of the magnet (i.e., the area z^2) at coil B. In this example, the extended Pythagoras Theorem using triangles not only tells us that the total electromagnetic intensity perceived by the system is not always constant, but also it quantifies how much of that intensity is wasted via the coupling term xy .

Exercise 2. *Imagine the magnet has a hypothetical intensity of 1, and coil A is recording an intensity $x^2=0.7^2=0.49$. How much intensity is being recorded by coil B located in a clockwise position at 120° , and how much is being wasted?*

Solution. Assume $z=1^2$ and $x^2=0.7^2=0.49$. Applying the extended Pythagoras Theorem using triangles gives

$$0.7^2 + (0.7)y + y^2 = 1^2$$

Re-arranging gives the quadratic function

$$y^2 + (0.7)y - 0.51 = 0$$

Finding the root gives

$$y = -\frac{0.7}{2} + \frac{1}{2}\sqrt{0.7^2 - 4(-0.51)} \approx 0.445$$

This means that coil A sees an intensity of $x^2=0.49$ or 49%, while coil B sees only $y^2=0.198$ or approximately 20%. For this particular angular position, the coupling part of the electromagnetic field generated by the magnet $xy=(0.7)(0.445)\approx 0.312$ or approximately 31%, is not passing by either coil A or B, and is lost in a “blind spot” in an angular location between them. This 31% represents a waste, as the turning of the

magnet is not fully electromagnetically inducing this sector of the system at this particular angular position. If one adds all the separate components, the result is the magnet peak intensity 1, or

$$x^2 + xy + y^2 = z^2 \\ \implies 0.49 + 0.312 + 0.198 = 1$$

To the operation of the present triple coil system disposed in a triangular manner, this unused electromagnetic induction capacity can be seen as a loss in efficiency, or simply a detriment that needs improving. The solution to this problem was introduced by Tesla that added an extra coil in between each of the original three, resulting in a hexagonal or dual-inverted triangle configuration [12]. In turn, this hexagonal contemporary alternating current generator/motor is well known for delivering continuous power generation (or loading in the case of a motor) at constant rating for every magnet rotation, thus being used in industry all around the world throughout the past century until the present today.

Another example of an application in Electrical Engineering is a cellular base antenna station deployed in an array along three faces, arranged in a triangular manner — with each face having typically 3 or 4 radiating elements (Figure 14a) [13], for which there are real examples (Figure 14b). The analysis done to the AC generator/motor is valid by replacing the rotating magnet with a transmitting mobile phone, and the coils by each face of the triangular antenna array. The directivity, and hence intensity of signal strength perceived by two of its three antenna array faces 1 and 2 disposed at 120 degrees to each other will vary as the phone rotates around the station (by the displacement of the individual, as it transits), recording the same reduction/increase in signal behavior as that explained before for the AC generator/motor application (where the magnet rotated at the center of two radially equidistant coils angular-spaced by 120 degrees).

Figure 14. Potential additional application of the extended Pythagoras Theorem to a cellular base antenna station with array faces deployed in a triangular manner: (a) illustrated and (b) actual

5 Conclusion

The Pythagoras Theorem, while providing a fundamental cornerstone to science, is not singular. Like all great theories, the Pythagoras Theorem pertains to a larger family of possibilities. In this article, the classical approach of relating areas with squares is moved into the new arena of relating areas with triangles, which implies moving away from the right-angled triangle - as the connecting element - to a 120-degree-angled triangle. Both theorems are proven and discussed back-to-back. The classical Pythagoras Theorem is explained first, establishing a particular approach that then serves as a baseline to evolve into the new triangle-based theorem. A key outcome was the identification of the coupling term xy as the main difference between these two theorems. Being explained for the first time geometrically and mathematically, the origin of this coupling term reaches out to a time period before trigonometric functions (like cosine), where the law of cosines was not yet known. The usefulness of both the Pythagoras Theorem and its variant is demonstrated with dedicated classroom exercises on both civil engineering (bridge building) and electrical engineering (alternating current generators/motors and triangular cellular base antenna stations), showing that one theorem complements the other side-by-side, hinting that both are actually particular cases of a more general rule.

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